MATH 105 101 Midterm 2 Sample 1 Solutions

1. (20 marks)

   (a) (5 marks) Find the derivative of the function:

$$F(x) = \int_1^{\ln(x)} \sqrt{4 + 2t + \sin(t)}\, dt,$$

at the point $x = 1$. **Simplify the answer.**

Solution: Using the Fundamental Theorem of Calculus part 1 and applying chain rule, we get:

$$\frac{dF}{dx} = \left(\sqrt{4 + 2\ln(x) + \sin(\ln x)}\right)\frac{1}{x}.$$

Thus, at $x = 1$,

$$\frac{dF}{dx}\,|_{x=1} = \left(\sqrt{4 + 2\ln 1 + \sin(\ln 1)}\right)\frac{1}{1} = \sqrt{4} = 2.$$

   (b) (5 marks) Use Simpson's Rule to approximate

$$\int_0^{\pi} \sin(x)\, dx$$

with $n = 4$ subintervals. **Simplify the answer.**

Solution: We have $a = 0$, $b = \pi$, $n = 4$, and $f(x) = \sin(x)$. So, $\Delta x = \frac{b-a}{n} = \frac{\pi}{4}$. There are 5 grid-points using the formula $x_k = a + k\Delta x$:

$$x_0 = 0, \quad x_1 = \frac{\pi}{4}, \quad x_2 = \frac{\pi}{2}, \quad x_3 = \frac{3\pi}{4}, \quad x_4 = \pi.$$

Using Simpson's Rule, we get:

$$\begin{aligned}
S_4 &= \frac{\Delta x}{3}(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)) \\
&= \frac{\pi}{12}\left(\sin 0 + 4\sin\left(\frac{\pi}{4}\right) + 2\sin\left(\frac{\pi}{2}\right) + 4\sin\left(\frac{3\pi}{4}\right) + \sin\pi\right) \\
&= \frac{\pi}{12}\left(0 + 4\left(\frac{\sqrt{2}}{2}\right) + 2(1) + 4\left(\frac{\sqrt{2}}{2}\right) + 0\right) \\
&= \frac{\pi}{12}(2 + 4\sqrt{2}) = \frac{\pi(1 + 2\sqrt{2})}{6}.
\end{aligned}$$

(c) (5 marks) Find the definite integral

$$\int_{-2}^{1} \frac{5}{x^3}\,dx.$$

Solution: Note that $\frac{5}{x^3}$ is discontinuous at $x = 0$ which is in the interval $[-2, 1]$, so this is an improper integral. We have:

$$\int_{-2}^{1} \frac{5}{x^3}\,dx = \lim_{a\to 0^-} \int_{-2}^{a} \frac{5}{x^3}\,dx + \lim_{b\to 0^+} \int_{b}^{1} \frac{5}{x^3}\,dx,$$

if both limits exist. First, we have that:

$$\int \frac{5}{x^3}\,dx = \int 5x^{-3}\,dx = -\frac{5}{2}x^{-2} + C.$$

So,

$$\begin{aligned} \lim_{a\to 0^-} \int_{-2}^{a} \frac{5}{x^3}\,dx &= \lim_{a\to 0^-} -\frac{5}{2}x^{-2} \,|_{-2}^{a} \\ &= \lim_{a\to 0^-} -\frac{5}{2a^2} + \frac{5}{8} = \infty. \end{aligned}$$

Since one of the limits does not exist, the improper integral diverges.

(d) (5 marks) Find the indefinite integral

$$\int x^2 \ln(x)\,dx.$$

Solution: Using integration by parts with $u = \ln(x)$, $dv = x^2\,dx$, we get $du = \frac{dx}{x}$ and $v = \frac{x^3}{3}$. So,

$$\begin{aligned} \int x^2 \ln(x)\,dx &= \frac{x^3}{3}\ln x - \int \frac{x^3}{3}\left(\frac{dx}{x}\right) \\ &= \frac{x^3}{3}\ln(x) - \frac{1}{3}\int x^2 dx \\ &= \frac{x^3}{3}\ln(x) - \frac{x^3}{9} + C. \end{aligned}$$

2. (10 marks)

(a) (8 marks) Compute the Left Riemann sum for $f(x) = x + 2$ on the interval $[-2, 4]$ using n equal subintervals. Use the summation identities to simplify the answer.

Solution: We have $a = -2$, $b = 4$, and $\Delta x = \frac{b-a}{n} = \frac{6}{n}$. For Left Riemann sum, we have:

$$x_k^* = x_{k-1} = a + (k-1)\Delta x = -2 + \frac{6(k-1)}{n},$$
$$f(x_k^*) = x_k^* + 2 = -2 + \frac{6(k-1)}{n} + 2 = \frac{6(k-1)}{n} = \frac{6k}{n} - \frac{6}{n}.$$

Then, The Left Riemann sum is:

$$\begin{aligned}\sum_{k=1}^{n} f(x_k^*)\Delta x &= \sum_{k=1}^{n} \left(\frac{6k}{n} - \frac{6}{n}\right)\frac{6}{n} \\ &= \sum_{k=1}^{n} (\frac{36k}{n^2} - \frac{36}{n^2}) = \sum_{k=1}^{n} \frac{36k}{n^2} - \sum_{k=1}^{n} \frac{36}{n^2}.\end{aligned}$$

Using the summation identities, we get:

$$\begin{aligned}\sum_{k=1}^{n} f(x_k^*)\Delta x &= \frac{36}{n^2}\sum_{k=1}^{n} k - \frac{36}{n^2}\sum_{k=1}^{n} 1 \\ &= \frac{36n(n+1)}{2n^2} - \frac{36n}{n^2} = \frac{18n^2 - 18n}{n^2}.\end{aligned}$$

(b) (2 marks) Use the answer in part (a) to evaluate $\int_{-2}^{4}(x+2)\,dx$. *An answer <u>without</u> making use of part (a) will be given zero marks.*

Solution: From the definition of definite integrals, we get:

$$\int_{-2}^{4}(x+2)\,dx = \lim_{n\to\infty} \frac{18n^2 - 18n}{n^2}.$$

To evaluate this, we divide the numerator and the denominator by n^2 and get:

$$\lim_{n\to\infty} \frac{18n^2 - 18n}{n^2} = \lim_{n\to\infty} \frac{18 - \frac{18}{n}}{1} = 18.$$

Thus, $\int_{-2}^{4}(x+2)\,dx = 18$.

3. (10 marks) Solve the initial value problem:

$$\frac{dy}{dt} = \frac{e^y(3t+11)}{t^2-t-6}, \qquad y(2) = 0.$$

You may leave the answer in its implicit form.

Solution: We have:

$$\frac{dy}{dt} = \frac{e^y(3t+11)}{t^2-t-6} \Leftrightarrow \frac{dy}{e^y} = \frac{3t+11}{t^2-t-6}\,dt.$$

Next, we want to integrate each side with respect to the respective variables. The left hand side yields:

$$\int \frac{dy}{e^y} = \int e^{-y}\,dy = -e^{-y} + C.$$

For the integral on the right hand side $\int \frac{3t+11}{t^2-t-6}\,dt$. We will use the method of partial fractions, since $t^2 - t - 6 = (t-3)(t+2)$. Set:

$$\begin{aligned} \frac{3t+11}{t^2-t-6} &= \frac{A}{t-3} + \frac{B}{t+2} \\ &= \frac{A(t+2) + B(t-3)}{(t+2)(t-3)} \\ &= \frac{(A+B)t + (2A-3B)}{t^2-t-6} \\ \Rightarrow 3t + 11 &= (A+B)t + (2A-3B). \end{aligned}$$

So, $A + B = 3$ and $2A - 3B = 11$. Since $A = 3 - B$, we get $11 = 2A - 3B = 2(3-B) - 3B = 6 - 5B$, which gives $B = -1$ and thus, $A = 4$. So,

$$\int \frac{3t+11}{t^2-t-6}\,dt = \int \frac{4}{t-3}\,dt - \int \frac{1}{t+2}\,dt = 4\ln|t-3| - \ln|t+2| + C.$$

Thus, $-e^{-y} = 4\ln|t-3| - \ln|t+2| + C$. To solve for C, we use $y(2) = 0$, which means:

$$-e^0 = 4\ln|2-3| - \ln|2+2| + C \Rightarrow C = -1 + \ln(4).$$

Hence,

$$-e^{-y} = 4\ln|t-3| - \ln|t+2| - 1 + \ln(4).$$

4. (10 marks) Evaluate the definite integral:

$$\int \frac{\sqrt{25x^2-4}}{x}\,dx.$$

Solution: We have that:

$$\int \frac{\sqrt{25x^2-4}}{x}\,dx = 5\int \frac{\sqrt{x^2-4/25}}{x}\,dx.$$

We use trigonometric substitution with $x = \frac{2}{5}\sec(\theta)$. Then,

$$\begin{aligned} dx &= \frac{2}{5}\sec(\theta)\tan(\theta)\,d\theta, \\ \sqrt{x^2-4/25} &= \sqrt{4/25\sec^2(\theta)-4/25} = \sqrt{4/25\tan^2(\theta)} = \frac{2}{5}\tan(\theta). \end{aligned}$$

So,

$$\begin{aligned} \int \frac{\sqrt{25x^2-4}}{x}\,dx &= 5\int \frac{2/5\tan(\theta)}{2/5\sec(\theta)}\,(2/5\sec(\theta)\tan(\theta)\,d\theta) \\ &= 2\int \tan^2(\theta)\,d\theta = 2\int (\sec^2(\theta)-1)\,d\theta \\ &= 2\int \sec^2(\theta)\,d\theta - 2\int d\theta \\ &= 2\tan(\theta) - 2\theta + C. \end{aligned}$$

Since $x = \frac{2}{5}\sec(\theta)$, we get $\frac{5x}{2} = \sec(\theta)$ so $\cos(\theta) = \frac{2}{5x}$. Hence, $\theta = \arccos\left(\frac{2}{5x}\right)$. Also, as computed above, $\sqrt{x^2-4/25} = \frac{2}{5}\tan(\theta)$, so we get:

$$\int \frac{\sqrt{25x^2-4}}{x}\,dx = 2\tan(\theta) - 2\theta + C = 5\sqrt{x^2-4/25} - 2\arccos\left(\frac{2}{5x}\right) + C.$$